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6. AUTHOR(S)
M. J. Potasek

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)
Department of Applied Physics
Columbia University
New York, New York 10027

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This research centers on both quantum and classical effects of femtosecond pulses.

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Report

A.) Statement of the Problem:

The research developed mathematical models of physical phenomena for the study of complex time-dependent, nonlinear partial differential equations of interest in nonlinear optics. Applications of the work include optical communications, optical switching and future optical computing.

The research involved the development of mathematical models describing femtosecond pulse propagation in nonlinear optical media. This research is significant because, as the duration of optical pulses shortens, new interactions arise. Recent advances in femtosecond light sources makes possible the study of new phenomena. These interactions must be described by novel nonlinear partial differential equations. A second important area of research involved the interaction of multiple beams of light for intensity dependent ultrafast optical switching. In particular, we have investigated novel solitons.

B.) Summary of Results:

Research in both quantum and classical areas has been investigated. We have obtained exact solutions to coupled higher-order nonlinear Schrödinger equations. This represents the first work in this significant area. These equations are then used to model femtosecond all-optical switching, which has important applications in the optical computing area.

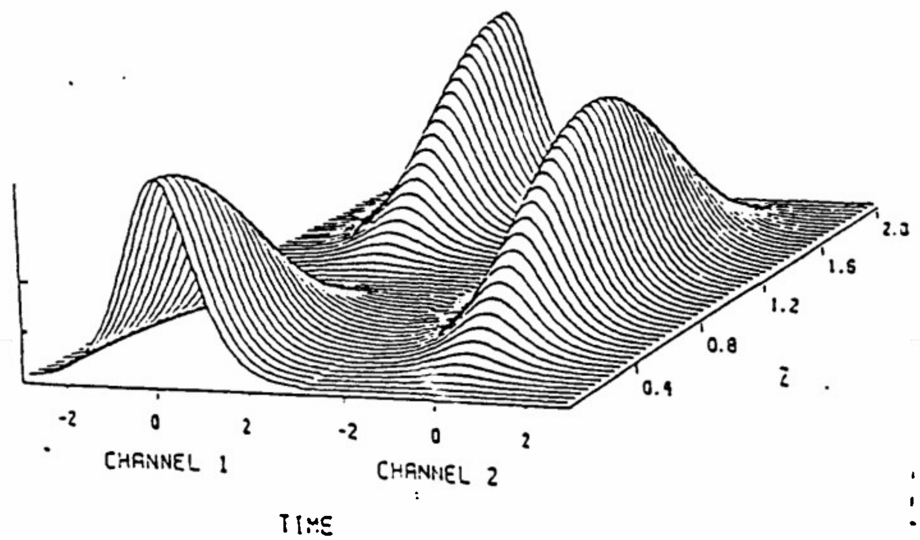
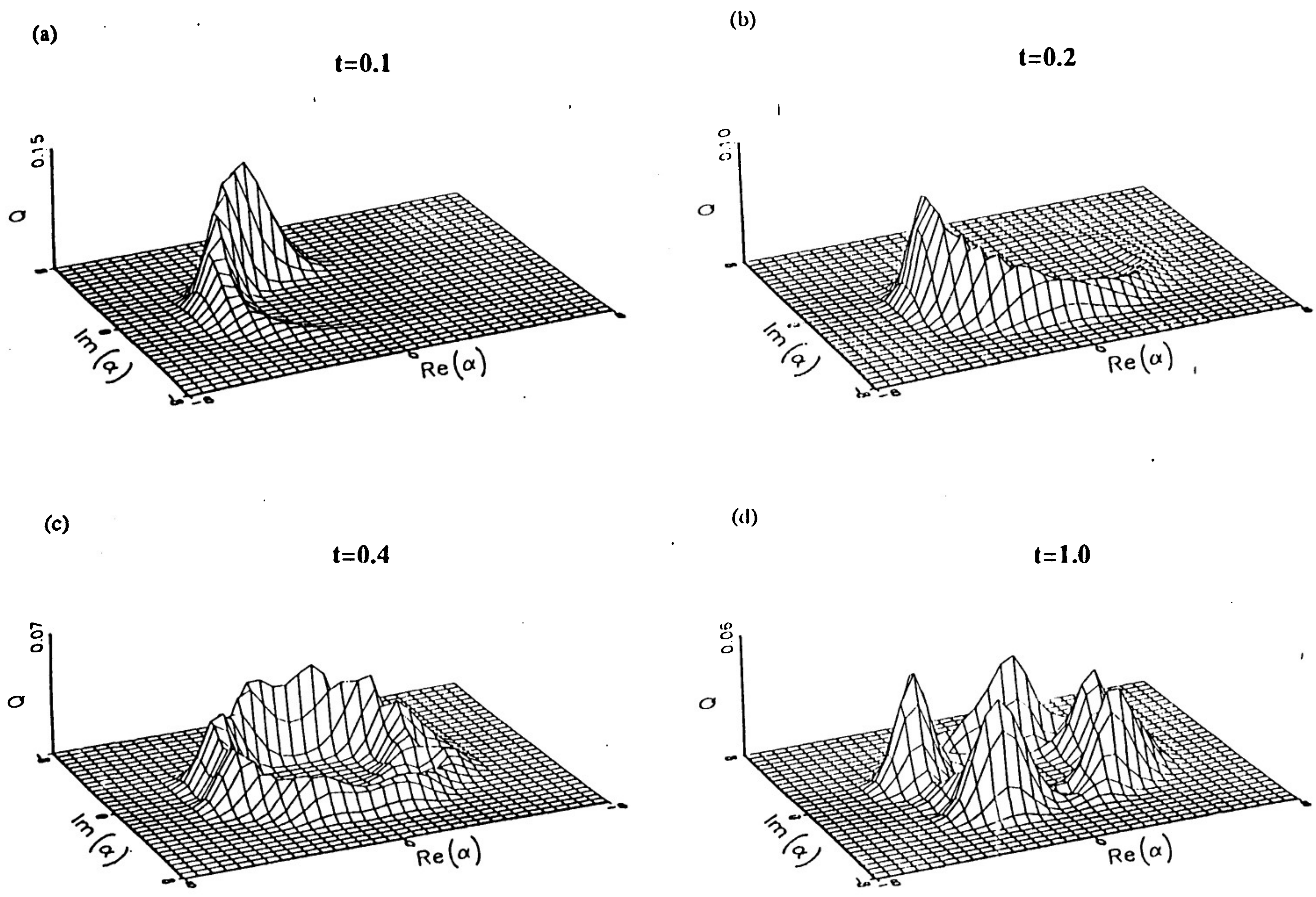


FIG. 1. Switching dynamics for light into one waveguide only, where the distance is in units of L_c and the time is in units of ω^{-1}.

Furthermore, using the time-dependent Harytree approximation, we have derived the first investigation of quantum effects for femtosecond pulses. This new work may lead to a greater understanding of quantum noise. Our results describe the propagation of femtosecond solitons in nonlinear optics. These solitons travel at velocities that differ from those of the picosecond solitons obtained from the standard quantized nonlinear Schrödinger equation. From the quantum solutions, we find that the soliton experiences phase spreading and self-squeezing as it propagates.



(a)
t=0.1
0.15
Q
Im(α)
Re(α)
(b)
t=0.2
0.10
Q
Im(α)
Re(α)
(c)
t=0.4
0.07
Q
Im(α)
Re(α)
(d)
t=1.0
0.05
Q
Im(α)
Re(α)

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"Femtosecond Solitons in Nonlinear Optical Fibers: Classical and Quantum Effects," Phys. Rev.A, submitted

"An Exact Solution for Femtosecond Pulses including the Effects of the Soliton Self-Frequency Shift," J. Math. Phys., submitted.

"Soliton Solutions to Coupled Higher-Order Nonlinear Schrödinger Equations," J. Math. Phys. **33**, 1208 (1992).

"Quantum Theory of Femtosecond Solitons in Optical Fib ers," Quantum Optics, accepted.

"Femtosecond Pulses in Directional Couplers near the Zero Dispersion Wavelength," Phys. Rev. A, submitted.

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M. J. Potasek

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None

Femtosecond Solitons in Nonlinear Optical Fibers: Classical and Quantum Effects

F. Singer, M. J. Potasek, J. M. Fang and M. C. Teich

Departments of Physics, Applied Physics, and Electrical Engineering

Columbia University

New York, New York 10027

Abstract

We use the time-dependent Hartree approximation to obtain solutions to a quantized higher-order nonlinear Schroedinger equation. This equation describes pulses propagating in nonlinear optical fibers and, under certain conditions, has femtosecond soliton solutions. These solitons travel at velocities that differ from those of the picosecond solitons obtained from the standard quantized nonlinear Schroedinger equation. Furthermore, we find that quadruple-clad fibers are required for the propagation of these solitons, unlike the solitons of the standard nonlinear Schroedinger equation which can propagate in graded-index optical fibers. From the quantum solution, we find that the soliton experiences phase-spreading and self-squeezing as it propagates.

Soliton solutions to coupled higher-order nonlinear Schrödinger equations

R. S. Tasgal and M. J. Potasek
Department of Applied Physics, Columbia University, New York, New York 10027

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A set of coupled higher-order nonlinear Schrödinger equations, which describe electromagnetic pulse propagation in coupled optical waveguides, is formulated in terms of an eigenvalue problem. Using that result, the inverse scattering problem is solved and explicit soliton solutions are found. Additionally, linear coupling terms are studied systematically.

I. INTRODUCTION

Optical signal processing is attracting interest because of its ultrafast response time. Currently, electro-optic devices generally require a cumbersome interface between electronics and optics. On the other hand, all optical signal processing utilizing only the nonlinear index of refraction results in partial loss of the signal due to nonuniform intensity distribution within the pulse. Solitons, and in certain cases solitary waves, which balance nonlinearity and dispersion, can improve system performance due to their remarkable stability properties.

Slowly varying electromagnetic waves in a nonlinear medium (an optical fiber, for example) are described by the nonlinear Schrödinger equation (NLS). In order to increase bit rates it is necessary to decrease the pulse width. As pulse lengths become comparable to the wavelength, however, the NLS equation becomes inadequate, as additional terms must now be considered. We will refer to equations which include these terms as higher-order nonlinear Schrödinger equations (HNLS).

One of the first HNLS equations to be solved exactly (by Hirota[1] in 1973, two years after the simple NLS equation was solved[2]) and, in a sense, the simplest is

$$iq_z + i\delta q_t + (\beta/2)q_{tt} + \beta|q|^2q - i\varepsilon(q_{ttt} + 2\mu|q|^2q_t) = 0, \tag{1.1}$$

where $\mu = 3$ and ε approaches zero when the pulse width is long compared to the wavelength.

There are several ways to generalize the HNLS Eq. (1.1) to a set of coupled equations, depending on the physical situation that is being modeled. A fairly general form of coupled HNLS equations is

$$\begin{aligned} &iq_{1z} + i(\delta_+ + \delta_-)q_{1t} + (\beta/2)q_{1tt} + \beta(|q_1|^2 \\ &\quad + \gamma|q_2|^2)q_1 + (\Delta_+ + \Delta_-)q_1 + (K_+ \\ &\quad + iK_-)q_2 - i\varepsilon[q_{1ttt} + \mu(|q_1|^2 \\ &\quad + \gamma|q_2|^2)q_{1t} + \mu(q_1^*q_{1t} + \gamma q_2^*q_{2t})q_1] = 0, \end{aligned} \tag{1.2}$$

$$\begin{aligned} &iq_{2z} + i(\delta_+ - \delta_-)q_{2t} + (\beta/2)q_{2tt} + \beta(\gamma|q_1|^2 \\ &\quad + |q_2|^2)q_2 + (\Delta_+ - \Delta_-)q_2 + (K_+ - iK_-)q_1 \\ &\quad - i\varepsilon[q_{2ttt} + \mu(\gamma|q_1|^2 + |q_2|^2)q_{2t} + \mu(\gamma q_1^*q_{1t} \\ &\quad + q_2^*q_{2t})q_2] = 0. \end{aligned} \tag{1.3}$$

A nonlinear directional coupler has $\delta_- = \Delta_+ = \Delta_- = K_- = 0$ and $K_+ \neq 0$.[3] A birefringent single mode fiber[4,5] and rocking fiber rotator,[6] in which the fiber is periodically twisted, have $\Delta_+ = K_- = 0$ and $\delta_-, \Delta_-, K_+ \neq 0$, where γ is a function of the ellipticity angle θ and two material parameters a and b:

$$\gamma = \frac{2a + 2b\sin^2\theta}{2a + b\cos^2\theta} \quad \text{(Ref.7)}$$

in optical fibers $a = b$.

Equations (1.2) and (1.3) with $\delta_- = \Delta_- = K_+ = K_- = \varepsilon = 0$ and $\Delta_+ \neq 0$ describe a nonrelativistic boson field.[8] In a weakly relativistic plasma, nonlinear coupling of two polarized transverse waves with dispersion is described by $\delta_+ = \delta_- = \Delta_+ = \Delta_- = K_+ = K_- = \varepsilon = 0$.[9] Also, for the case $\delta_- = \beta = \Delta_+ = \Delta_- = K_+ = K_- = 0$ with $q_1, q_2 \in R$, Eqs. (1.2) and (1.3) are a pair of coupled modified Korteweg–de Vries equations. The intermode switching term K_-, which emerges in a natural way from the mathematical derivation below, has not been considered in previous soliton work.

Solitons have been found in a variety of (uncoupled) higher-order NLS equations. Analytic solutions to the simplest NLS equation—Eq. (1.1) in the limit $\varepsilon \to 0$—were discovered in 1971 by Zakharov and Shabat.[2,10] Hirota[1] obtained exact soliton solutions to the HNLS Eq. (1.1) by transforming the NLS equation into a homogeneous form of the second degree. (While this approach produces several valuable insights, it has the disadvantages of being *ad hoc*, somewhat hard to work with, and it treats the higher-order terms and NLS terms differ-

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ently.) Sasa and Satsuma recently discovered soliton solutions to a more complex HNLS equation.[11] The derivative[12,13] and mixed derivative[14,15] NLS equations have been solved. Painlevé techniques produce other solutions.[16] Some exact soliton solutions may be found by transformation to known NLS equations.[17]

Exact solitary waves (which may or may not be solitons) can be found by direct substitution.[18] Approximate solitary waves can be found by various techniques. Among the most useful are numerical computation[19] and variational methods.[20] (Sometimes, when the exact solutions are not known, these are also fruitful approaches to integrable systems.[21,22])

Recently the coupled NLS equations, without higher order terms, have been the focus of intense attention.[23] Manakov[24] found explicit soliton solutions to the coupled NLS Eqs. (1.2) and (1.3) with equal self- and cross-phase modulation, but without either higher-order terms or linear coupling $\delta_-=\Delta_+=\Delta_-= K_+=K_-=\varepsilon=0$, $\gamma=1$. Elphick[8] used the quantum inverse scattering method to study the Manakov equations with an added symmetric self coupling term $\Delta_+\neq 0$, $\delta_-=\Delta_-=K_+=K_-=\varepsilon=0$, $\gamma=1$. Belanger and Paré[25] found a transformation that reduces a set of coupled NLS equations with symmetric linear cross-coupling terms $K_+\neq 0$, $\delta_-=\Delta_+= \Delta_-=K_-=\varepsilon=0$, $\gamma=1$ to the Manakov case, thereby finding solitons with periodic energy exchange between the two coupled modes.

Solitary waves in the coupled NLS equations, including nonintegrable (and consequently soliton destroying) terms such as $\delta_-\neq 0$, $\gamma\neq 1$, and absorption, are also of interest. We mention only a sampling of work in this field, since it is too vast for a thorough survey here. Christodoulides and Joseph[26] discovered exact vector solitons in coupled NLS equations with a birefringence term. Paré and Florjanczak[27] found analytic solutions using a Lagrangian variational method. Stability analyses have been performed.[28,29] There is also a large amount of numerical work.[30-35]

To date there has been no work on coupled nonlinear Schrödinger equations with higher-order terms, and linear coupling terms have not been studied systematically. In this paper, using the method of Ablowitz, Kaup, Newell, and Segur[36] (AKNS), we formulate the coupled NLS equations in a more systematic way than has been done previously. With that result, the coupled NLS equations are generalized in a very natural way to include higher-order terms and other new linear coupling terms. The inverse scattering transform[37,38] is then straightforwardly carried out, yielding explicit solutions.

II. FORMULATION OF THE EIGENVALUE PROBLEM

The method of AKNS is begun by writing the not yet fully defined eigenvalue problem

$$v_t=Tv,$$

$$T\equiv\begin{pmatrix} -i\rho & q_1 & q_2 \\ -q_1^* & i\rho & 0 \\ -q_2^* & 0 & i\rho \end{pmatrix}, \tag{2.1}$$

$$v_z=Zv,$$

$$Z_{ij}=\sum_{n=0}^{N} Z_{ij}^{(n)}\rho^n. \tag{2.2}$$

The integrability condition for Eqs. (2.1)–(2.2) is

$$T_z-Z_t+[T,Z]=0. \tag{2.3}$$

Writing each of the nine components of the matrix explicitly and matching terms of the same order in ρ yields an iterative method of determining Z:

$$Z_{12}^{(n)}=(i/2)(Z_{12t}^{(n+1)}+(Z_{11}^{(n+1)}-Z_{22}^{(n+1)})q_1 - Z_{32}^{(n+1)}q_2), \tag{2.4}$$

$$Z_{13}^{(n)}=(i/2)(Z_{13t}^{(n+1)}+(Z_{11}^{(n+1)}-Z_{33}^{(n+1)})q_2 - Z_{23}^{(n+1)}q_1), \tag{2.5}$$

$$Z_{21}^{(n)}=-(i/2)(Z_{21t}^{(n+1)}+(Z_{11}^{(n+1)}-Z_{22}^{(n+1)})q_1^* - Z_{23}^{(N+1)}q_2^*), \tag{2.6}$$

$$Z_{31}^{(n)}=-(i/2)(Z_{31t}^{(n+1)}+(Z_{11}^{(n+1)}-Z_{33}^{(n+1)})q_2^* - Z_{32}^{(n+1)}q_1^*), \tag{2.7}$$

$$Z_{11t}^{(n)}=Z_{21}^{(n)}q_1+Z_{31}^{(n)}q_2+Z_{12}^{(n)}q_1^*+Z_{13}^{(n)}q_2^*, \tag{2.8}$$

$$Z_{22t}^{(n)}=-Z_{21}^{(n)}q_1-Z_{12}^{(n)}q_1^*, \tag{2.9}$$

$$Z_{33t}^{(n)}=-Z_{31}^{(n)}q_2-Z_{13}^{(n)}q_2^*, \tag{2.10}$$

$$Z_{23t}^{(n)}=-Z_{21}^{(n)}q_2-Z_{13}^{(n)}q_1^*, \tag{2.11}$$

$$Z_{32t}^{(n)}=-Z_{31}^{(n)}q_1-Z_{12}^{(n)}q_2^*, \tag{2.12}$$

and also four equalities in the zeroth order (1,2), (1,3), (2,1), and (3,1) matrix components of the integrability condition, Eq. (2.3):

$$q_{1z}-Z_{12t}^{(0)}-(Z_{11}^{(0)}-Z_{22}^{(0)})q_1+Z_{32}^{(0)}q_2=0, \tag{2.13}$$

$$q_{2z}-Z_{13t}^{(0)}-(Z_{11}^{(0)}-Z_{33}^{(0)})q_2+Z_{23}^{(0)}q_1=0, \tag{2.14}$$

$$q_{1z}^* + Z_{21t}^{(0)} + (Z_{11}^{(0)} - Z_{22}^{(0)})q_1^* - Z_{23}^{(0)}q_2^* = 0, \tag{2.15}$$

$$q_{2z}^* + Z_{31t}^{(0)} + (Z_{11}^{(0)} - Z_{33}^{(0)})q_2^* - Z_{32}^{(0)}q_1^* = 0. \tag{2.16}$$

Each iteration allows five constants of integration in $Z^{(n)}$. A constant times the identify in Z or T does not affect the integrability condition, Eq. (2.3). There remain four possible *physical* degrees of freedom for each term in the polynomial Z.

Setting $Z^{(4)} = 0$, an appropriate choice of the constants of integration and trace yields

$$Z^{(3)} = -8i\varepsilon\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{2.17}$$

$$Z^{(2)} = -4\varepsilon\begin{pmatrix} 0 & q_1 & q_2 \\ -q_1^* & 0 & 0 \\ -q_2^* & 0 & 0 \end{pmatrix} + 4i\left(\frac{\beta}{2}\right)\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{2.18}$$

$$Z^{(1)} = -2i\varepsilon\begin{pmatrix} |q_1|^2 + |q_2|^2 & q_{1t} & q_{2t} \\ q_{1t}^* & -|q_1|^2 & -q_2q_1^* \\ q_{2t}^* & -q_1q_2^* & -|q_2|^2 \end{pmatrix} + 2\left(\frac{\beta}{2}\right)\begin{pmatrix} 0 & q_1 & q_2 \\ -q_1^* & 0 & 0 \\ -q_2^* & 0 & 0 \end{pmatrix}, \tag{2.19}$$

$$Z^{(0)} = \varepsilon\begin{pmatrix} q_{1t}q_1^* - q_1q_{1t}^* + q_{2t}q_2^* - q_2q_{2t}^* & q_{1tt} + 2(|q_1|^2 + |q_2|^2)q_1 & q_{2tt} + 2(|q_1|^2 + |q_2|^2)q_2 \\ -q_{1tt}^* - 2(|q_1|^2 + |q_2|^2)q_1^* & -(q_{1t}q_1^* - q_1q_{1t}^*) & -(q_{2t}q_1^* - q_2q_{1t}^*) \\ -q_{2tt}^* - 2(|q_1|^2 + |q_2|^2)q_2^* & -(q_{1t}q_2^* - q_1q_{2t}^*) & -(q_{2t}q_2^* - q_2q_{2t}^*) \end{pmatrix}$$
$$+ i\left(\frac{\beta}{2}\right)\begin{pmatrix} |q_1|^2 + |q_2|^2 & q_{1t} & q_{2t} \\ q_{1t}^* & -|q_1|^2 & -q_2q_1^* \\ q_{2t}^* & -q_1q_2^* & -|q_2|^2 \end{pmatrix} - i\begin{pmatrix} 0 & 0 & 0 \\ 0 & \Delta_+ + \Delta_- & K_+ - iK_- \\ 0 & K_+ + iK_- & \Delta_+ - \Delta_- \end{pmatrix}. \tag{2.20}$$

We have neglected the symmetric group velocity term δ_+, which enters at the $Z^{(1)}$level, since it can be eliminated by a trivial change of variables. Setting Eq. (2.13) equivalent to Eq. (2.15) and Eq. (2.14) equivalent to Eq. (2.16) forces β, Δ_+, Δ_-, K_+, K_-, $\varepsilon \in \mathrm{R}$. Insertion of $Z^{(0)}$ into Eqs. (2.13)–(2.16) yields a set of coupled HNLS equations:

$$iq_{1z} + (\beta/2)q_{1tt} + \beta(|q_1|^2 + |q_2|^2)q_1 + (\Delta_+ + \Delta_-)q_1 + (K_+ + iK_-)q_2 - i\varepsilon[q_{1ttt} + 3(|q_1|^2 + |q_2|^2)q_{1t} + 3(q_1^*q_{1t} + q_2^*q_{2t})q_1] = 0, \tag{2.21}$$

$$iq_{2z} + (\beta/2)q_{2tt} + \beta(|q_1|^2 + |q_2|^2)q_2 + (\Delta_+ - \Delta_-)q_2 + (K_+ - iK_-)q_1 - i\varepsilon[q_{2ttt} + 3(|q_1|^2 + |q_2|^2)q_{2t} + 3(q_1^*q_{1t} + q_2^*q_{2t})q_2] = 0. \tag{2.22}$$

These are Eqs. (1.2) and (1.3) with $\delta_+ = \delta_- = 0$, $\gamma = 1$, and $\mu = 3$. There are four constants of integration introduced in $Z^{(0)}$, two of them on the diagonal. Equations (2.21) and (2.22) therefore contain the most general linear coupling terms that the AKNS formalism allows for Eqs. (2.1)–(2.3).

III. ELIMINATION OF THE LINEAR COUPLING TERMS

Having formulated the coupled HNLS equations as above, the zeroth-order constants of integration may be diagonalized by a rotation:

$$\Lambda v_t = \Lambda T \Lambda^{-1} \Lambda v, \tag{3.1}$$

$$\Lambda v_z = \Lambda Z \Lambda^{-1} \Lambda v, \tag{3.2}$$

where

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-(i/2)\varphi}\cos(\theta/2) & e^{(i/2)\varphi}\sin(\theta/2) \\ 0 & -e^{-(i/2)\varphi}\sin(\theta/2) & e^{(i/2)\varphi}\cos(\theta/2) \end{pmatrix}, \tag{3.3}$$

$$\tan(\varphi)=\frac{K_-}{K_+}, \tag{3.4}$$

$$\tan(\theta)=\sqrt{K_+^2+K_-^2}/\Delta_-. \tag{3.5}$$

This is equivalent to the transformation

$$q_1=e^{(i/2)\varphi}[\cos(\theta/2)q_1'-\sin(\theta/2)q_2'], \tag{3.6}$$

$$q_1=e^{-(i/2)\varphi}[\sin(\theta/2)q_1'+\cos(\theta/2)q_2'], \tag{3.7}$$

with primed terms

$$K'_+=K'_-=0, \tag{3.8}$$

$$_-=\pm\sqrt{\Delta_-^2+K_+^2+K_-^2},\quad \operatorname{sign}(\Delta'_-)=\operatorname{sign}(\Delta_-), \tag{3.9}$$

and all the other terms unchanged.

The linear self coupling terms Δ_+ and Δ'_- may now be removed by a second substitution:

$$q_1'=e^{i(\Delta_++\Delta'_-)z}q_1'', \tag{3.10}$$

$$q_2'=e^{i(\Delta_+-\Delta'_-)z}q_2''. \tag{3.11}$$

That leaves the coupled HNLS equations (omitting the primes)

$$iq_{1z}+(\beta/2)q_{1tt}+\beta(|q_1|^2+|q_2|^2)q_1-i\varepsilon[q_{1ttt}$$
$$+3(|q_1|^2+|q_2|^2)q_{1t}+3(q_1^*q_{1t}+q_2^*q_{2t})q_1]=0, \tag{3.12}$$

$$iq_{2z}+(\beta/2)q_{2tt}+\beta(|q_1|^2+|q_2|^2)q_2-i\varepsilon[q_{2ttt}$$
$$+3(|q_1|^2+|q_2|^2)q_{2t}+3(q_1^*q_{1t}+q_2^*q_{2t})q_2]=0, \tag{3.13}$$

which, in the limit $\varepsilon\to 0$, is the Manakov case.

Bélanger and Paré[25] made the substitution

$$q_1=\cos(K_+z)q_1'-i\sin(K_+z)q_2', \tag{3.14}$$

$$q_2=-i\sin(K_+z)q_1'+\cos(K_+z)q_2' \tag{3.15}$$

to eliminate K_+ from the coupled NLS equations (1.2) and (1.3) with $\delta_-=\Delta_+=\Delta_-=K_-=\varepsilon=0$ and $\gamma=1$. In contrast to the transformation given by Eqs. (3.6)–(3.11), Belanger and Paré's transformation breaks down (i.e., fails to eliminate K_+) if $\Delta_-\neq 0$ or $K_-\neq 0$, although it does work for the other linear coupling terms and the higher-order terms. Equations (3.14) and (3.15) cannot be applied, for example, to a periodically twisted birefringent fiber. Neither transformation works if $\delta_-\neq 0$ or $\gamma\neq 1$.

IV. THE INVERSE SCATTERING TRANSFORM

Solitons solutions to the coupled HNLS equations may now be found using the inverse scattering transform. If $|q_1|$, $|q_2|\to 0$ as $|t|\to\infty$ (which implies bright solitons), then the Jost functions may be defined as the eigenfunctions v in Eqs. (2.1) and (2.2) with boundary conditions

$$\psi_{l1}\to\begin{pmatrix}1\\0\\0\end{pmatrix}e^{-i\rho t}\quad\text{as } t\to-\infty, \tag{4.1}$$

$$\psi_{l2}\to\begin{pmatrix}0\\1\\0\end{pmatrix}e^{i\rho t}\quad\text{as } t\to-\infty, \tag{4.2}$$

$$\psi_{l3}\to\begin{pmatrix}0\\0\\1\end{pmatrix}e^{i\rho t}\quad\text{as } t\to-\infty, \tag{4.3}$$

$$\psi_{r1}\to\begin{pmatrix}1\\0\\0\end{pmatrix}e^{-i\rho t}\quad\text{as } t\to\infty, \tag{4.4}$$

$$\psi_{r2}\to\begin{pmatrix}0\\1\\0\end{pmatrix}e^{i\rho t}\quad\text{as } t\to\infty, \tag{4.5}$$

$$\psi_{r3}\to\begin{pmatrix}0\\0\\1\end{pmatrix}e^{i\rho t}\quad\text{as } t\to\infty. \tag{4.6}$$

For $\rho\in R$, $T^\dagger=-T$. Therefore

$$\frac{\partial}{\partial t}[\psi_1^\dagger(\rho,t)\psi_2(\rho,t)]=0, \tag{4.7}$$

where $\psi_i(\rho,t)$ is a solution to Eqs. (2.1)–(2.3), and

$$\psi_{ri}^\dagger(\rho,t)\psi_{rj}(\rho,t)=\psi_{li}^\dagger(\rho,t)\psi_{lj}(\rho,t)=\delta_{ij}. \tag{4.8}$$

The Jost functions (4.1)–(4.3) are related to the Jost functions (4.4)–(4.6) by the scattering matrix α:

$$\psi_{li}(\rho,t)=\sum_{j=1}^{3}\alpha_{ij}(\rho)\psi_{rj}(\rho,t), \tag{4.9}$$

$$\alpha_{ij}(\rho)=\psi_{ri}^{*}(\rho,t)\psi_{lj}(\rho,t), \tag{4.10}$$

$$\sum_{k=1}^{3}\alpha_{ik}^{*}(\rho)\alpha_{jk}(\rho)=\delta_{ij}. \tag{4.11}$$

Using Eq. (4.10), $\alpha_{11}(\rho)$ may be analytically continued into the upper half-plane $\mathrm{Im}(\rho)>0$; and $\alpha_{22}(\rho)$, $\alpha_{23}(\rho)$, $\alpha_{32}(\rho)$, and $\alpha_{33}(\rho)$ into the lower half-plane $\mathrm{Im}(\rho)<0$. From this and the unitarity of α (4.11)

$$\alpha_{11}^{*}(\rho^{*})=\det\begin{pmatrix}\alpha_{22}(\rho) & \alpha_{23}(\rho)\\ \alpha_{32}(\rho) & \alpha_{33}(\rho)\end{pmatrix}. \tag{4.12}$$

At this point we posit that $\alpha_{11}(\rho)$ has N simple zeroes at the points $\rho_1,\rho_1,...,\rho_N$ in the upper half-plane. It will be shown below that the locations of the zeroes determine (some of) the physical parameters of the solitons.

Introduce an integral representation of the Jost functions

$$\psi_{r1}(\rho,t)=\begin{pmatrix}1\\0\\0\end{pmatrix}e^{-i\rho t}+\int_t^{\infty}K_{r1}(t,s)e^{-i\rho s}\,ds, \tag{4.13}$$

$$\psi_{r2}(\rho,t)=\begin{pmatrix}0\\1\\0\end{pmatrix}e^{i\rho t}+\int_t^{\infty}K_{r2}(t,s)e^{i\rho s}\,ds, \tag{4.14}$$

$$\psi_{r3}(\rho,t)=\begin{pmatrix}0\\0\\1\end{pmatrix}e^{i\rho t}+\int_t^{\infty}K_{r3}(t,s)e^{i\rho s}\,ds, \tag{4.15}$$

$$K_n=\begin{pmatrix}K_n^{(1)}\\K_n^{(2)}\\K_n^{(3)}\end{pmatrix}.$$

The functions q_1 and q_2 are found in terms of K by substitution of Eqs. (4.14) and (4.15) into Eq. (2.1):

$$q_1=-2K_{r2}^{(1)}(t,t), \tag{4.16}$$

$$q_2=-2K_{r3}^{(1)}(t,t). \tag{4.17}$$

To find $K_{r2}^{(1)}$ and $K_{r3}^{(1)}$ we first recall the definition of the scattering matrix (4.9):

$$\psi_{l1}(\rho,t)=\alpha_{11}(\rho)\psi_{r1}(\rho,t)+\alpha_{12}(\rho)\psi_{r2}(\rho,t)+\alpha_{13}(\rho)\psi_{r3}(\rho,t), \tag{4.18}$$

$$\psi_{l2}(\rho,t)=\alpha_{2}\ (\rho)\psi_{r1}(\rho,t)+\alpha_{22}(\rho)\psi_{r2}(\rho,t)+\alpha_{23}(\rho)\psi_{r3}(\rho,t), \tag{4.19}$$

$$\psi_{l3}(\rho,t)=\alpha_{31}(\rho)\psi_{r1}(\rho,t)+\alpha_{32}(\rho)\psi_{r2}(\rho,t)+\alpha_{33}(\rho)\psi_{r3}(\rho,t). \tag{4.20}$$

Substitute the integral representation of the Jost functions (4.13)–(4.15) into Eqs. (4.18)–(4.20). Operate on Eq. (4.18) with

$$\frac{1}{2\pi}\int_{C_+}d\rho\, e^{i\rho r}\frac{1}{\alpha_{11}(\rho)},$$

where C_+ goes from $-\infty$ to ∞, over $\rho_1...,\rho_N$ and on Eqs. (4.19) and (4.20) with

$$\frac{1}{2\pi}\int_{C_-}d\rho\, e^{-i\rho r}\frac{1}{\alpha_{11}^{*}(\rho^{*})},$$

where C_- goes from $-\infty$ to ∞, under $\rho_1^{*},...,\rho_N^{*}$.

This gives the Gel'fand, Levitan, and Marchenko (GLM) equations

$$0=K_{r1}^{(1)}(t,r)+\int_0^{\infty}ds[K_{r2}^{(1)}(t,t+s)F_{12}(t+s+r)+K_{r3}^{(1)}(t,t+s)F_{13}(t+s+r)], \tag{4.21}$$

$$0=K_{r2}^{(1)}(t,r)+F_{21}(t+s)+\int_0^{\infty}ds[K_{r1}^{(1)}(t,t+s)\times F_{21}(t+s+r)+K_{r3}^{(1)}(t,t+s)F_{23}(t+s-r)], \tag{4.22}$$

$$0=K_{r3}^{(1)}(t,r)+F_{31}(t+s)+\int_0^{\infty}ds[K_{r1}^{(1)}(t,t+s)\times F_{31}(t+s+r)+K_{r2}^{(1)}(t,t+s)F_{32}(t+s-r)], \tag{4.23}$$

where

$$F_{1j}(t)=\frac{1}{2\pi}\int_{C_+}d\rho\frac{\alpha_{1j}(\rho)}{\alpha_{11}(\rho)}e^{i\rho t}=-i\sum_{n=1}^{N}C_{1j}(z,\rho_n)e^{i\rho_n t}+\mathscr{F}\left[\frac{\alpha_{1j}(\rho)}{\alpha_{11}(\rho)}\right](t),$$

$$F_{ij}(t)=\frac{1}{2\pi}\int_{C_-} d\rho \frac{\alpha_{ij}(\rho)}{\alpha_{11}^*(\rho^*)} e^{-i\rho t}$$

$$=-i\sum_{n=1}^{N} C_{ij}(z,\rho_n^*)e^{-i\rho_n^* t}+\mathscr{F}\left[\frac{\alpha_{ij}(\rho)}{\alpha_{11}^*(\rho^*)}\right](-t),$$

$i=2,3.$

Note: the z dependence, which was previously implicit in α, is now written explicitly in C_{ij}.

The residues' contribution to F_{ij} corresponds to solitons; the Fourier transform part corresponds to radiation. Since we wish to obtain soliton solutions, we will neglect the latter. Performing the integrations, Eqs. (4.21)–(4.23) become

$$0=K_{r1}^{(1)}(t,r)-i\sum_{n=1}^{N} e^{i\rho_n(t+r)}[C_{12}(z,\rho_n)\hat{K}_{r2}(t,\rho_n)+C_{13}(z,\rho_n)\hat{K}_{r3}(t,\rho_n)], \tag{4.24}$$

$$0=K_{r2}^{(1)}(t,r)-i\sum_{n=1}^{N} e^{-i\rho_n^*(t+r)}C_{21}(z,\rho_n^*)\times(1+\hat{K}_{r1}(t,-\rho_n^*)), \tag{4.25}$$

$$0=K_{r3}^{(1)}(t,r)-i\sum_{n=1}^{N} e^{-i\rho_n^*(t+r)}C_{31}(z,\rho_n^*)\times(1+\hat{K}_{r1}(t,-\rho_n^*)), \tag{4.26}$$

where a hat denotes the operator

$$\hat{A}(t,\rho)\equiv\int_0^\infty ds\, e^{-i\rho s}A^{(1)}(t,t+s).$$

To find q_1 and q_2, set $r=t$ in Eqs. (4.25) and (4.26):

$$q_1=-2K_{r2}^{(1)}(t,t)=-2i\sum_{n=1}^{N} e^{-2i\rho_n^* t}C_{21}(z,\rho_n^*)(1+\hat{K}_{r1}(t,-\rho_n^*)), \tag{4.27}$$

$$q_2=-2K_{r3}^{(1)}(t,t)=-2i\sum_{n=1}^{N} e^{-2i\rho_n^* t}C_{31}(z,\rho_n^*)(1+\hat{K}_{r1}(t,-\rho_n^*)). \tag{4.28}$$

Now return to Eqs. (4.24)–(4.26). Substitute $r=t+s$ and operate with

$$\int_0^\infty ds\, e^{i\rho s}$$

to give a set of $3N$ linear equations in $1+\hat{K}_{r1}(t,-\rho_m^*)$, $\hat{K}_{r2}(t,\rho_m)$, and $\hat{K}_{r3}(t,\rho_m)$, with $m=1,\dots,N$:

$$1=1+\hat{K}_{r1}(t,-\rho_m^*)-\sum_{n=1}^{N}\frac{e^{2i\rho_n t}}{\rho_m^*-\rho_n}[C_{12}(z,\rho_n)\hat{K}_{r2}(t,\rho_n)+C_{13}(z,\rho_n)\hat{K}_{r3}(t,\rho_n)], \tag{4.29}$$

$$0=\hat{K}_{r2}(t,\rho_m)+\sum_{n=1}^{N}\frac{e^{-2i\rho_n^* t}}{\rho_m-\rho_n^*}C_{21}(z,\rho_n^*)\times(1+\hat{K}_{r1}(t,-\rho_n^*)), \tag{4.30}$$

$$0=\hat{K}_{r3}(t,\rho_m)+\sum_{n=1}^{N}\frac{e^{-2i\rho_n^* t}}{\rho_m-\rho_n^*}C_{31}(z,\rho_n^*)\times(1+\hat{K}_{r1}(t,-\rho_n^*)). \tag{4.31}$$

The z dependence of the Jost functions (4.1)–(4.6) may be determined from Eq. (2.2). For simplicity, we use the z dependence of the Jost functions in the limit $|t|\to 0$ to determine the z dependence of α_{ij} and, consequently, C_{ij} (for all t):

$$\begin{aligned}
&C_{12}\left(z,\frac{\xi}{2}+i\frac{\eta}{2}\right)\\
&\quad\propto C_{13}\left(z,\frac{\xi}{2}+i\frac{\eta}{2}\right)\propto C_{21}^*\left(z,\frac{\xi}{2}-i\frac{\eta}{2}\right)\propto C_{31}^*\left(z,\frac{\xi}{2}-i\frac{\eta}{2}\right)\\
&\quad\propto\exp\{i(\beta/2)(\xi+i\eta)^2-\varepsilon(\xi+i\eta)^3]z\}\\
&\quad\propto\exp\{\eta[(\beta/2)(2\xi)-\varepsilon(3\xi^2-\eta^2)]z\}\\
&\qquad\times\exp\{i[(\beta/2)(\xi^2-\eta^2)]z-\varepsilon\xi(\xi^2-3\eta^2)\}.
\end{aligned} \tag{4.32}$$

Substitution of the z dependence of the C_{ij}'s, Eq. (4.32) into Eqs. (4.29)–(4.31), and substitution, in turn, of the appropriate results of Eqs. (4.29)–(4.31) into Eqs. (4.27)–(4.28) gives exact soliton solutions.

V. THE ONE-SOLITON SOLUTION

The simplest nontrivial soliton, found by setting $N=1$ above, is a single solitary wave:

$$q_1''=\frac{-2ie^{-2i\rho^*t}C_{21}(\rho^*)}{1-[e^{2i(\rho-\rho^*)t}/(\rho-\rho^*)^2][C_{12}(\rho)C_{21}(\rho^*)+C_{13}(\rho)C_{31}(\rho^*)]},\tag{5.1}$$

$$q_2''=\frac{-2ie^{-2i\rho^*t}C_{31}(\rho^*)}{1-[e^{2i(\rho-\rho^*)t}/(\rho-\rho^*)^2][C_{12}(\rho)C_{21}(\rho^*)+C_{13}(\rho)C_{31}(\rho^*)]}.\tag{5.2}$$

On substitution on the z dependence Eq. (4.32) and $2\rho=\xi+i\eta$, and some algebra, we may express q_1 and q_2 in the form

$$q_1''=\sin(\alpha)e^{i\phi}-q'',\tag{5.3}$$

$$q_2''=\cos(\alpha)e^{-i\phi}-q'',\tag{5.4}$$

where

$$q''=\eta\ \mathrm{sech}\{\eta\{t-t_0+[(\beta/2)(2\xi)-\varepsilon(3\xi^2-\eta^2)]z\}\}$$
$$\times\exp\{-i\{\xi(t-t_0)+[(\beta/2)(\xi^2-\eta^2)$$
$$-\varepsilon\xi(\xi^2-3\eta^2)]z+\phi_+\}\}\tag{5.5}$$

$$\tan(\alpha)=|C_{12}(0,\rho)|/|C_{13}(0,\rho)|.\tag{5.6}$$

Finally, the transformation given by Eqs. (3.6)–(3.9) gives the unprimed q_1 and q_2:

$$q_1=e^{(i/2)\varphi}[\cos(\theta/2)\sin(\alpha)e^{i(\Delta'_-z+\phi_-)}$$
$$-\sin(\theta/2)\cos(\alpha)e^{-i(\Delta'_-z+\phi_-)}]q,\tag{5.7}$$

$$q_2=e^{-(i/2)\varphi}[\sin(\theta/2)\sin(\alpha)e^{i(\Delta'_-z+\phi_-)}$$
$$+\cos(\theta/2)\cos(\alpha)e^{-i(\Delta'_-z+\phi_-)}]q,\tag{5.8}$$

where

$$q=\eta\ \mathrm{sech}\{\eta\{t-t_0+[(\beta/2)(2\xi)-\varepsilon(3\xi^2-\eta^2)]z\}\}$$
$$\times\exp\{-i\{\xi(t-t_0)+[-\Delta_++(\beta/2)(\xi^2-\eta^2)$$
$$-\varepsilon\xi(\xi^2-3\eta^2)]z+\phi_+\}\};\tag{5.9}$$

α, ϕ_+, ϕ_-, and t_0 are free (real) parameters, as are the components of the eigenvalue $\xi/2$ and $\eta/2$. Recall that $\tan(\varphi)=K_-/K_+$,

$$\tan(\theta)=\sqrt{K_+^2+K_-^2}/\Delta_-,$$

and

$$\Delta'_-=\pm\sqrt{\Delta_-^2+K_+^2+K_-^2}.$$

The other parameters are given in the coupled HNLS equations.

In conclusion, we have obtained bright soliton solutions to a generalized set of coupled higher order nonlinear Schrödinger equations. Higher-order and NLS terms are treated the same way. Also, we found a transformation that eliminates all of the four linear coupling terms that the AKNS formalism allows for this problem. Future papers will focus on dark soliton solutions and mixed dark and bright soliton solutions.

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Letter to the Editor

LETTER TO THE EDITOR

Quantum theory of femtosecond solitons in optical fibres

F Singer, M J Potasek and M C Teich

Departments of Physics, Applied Physics and Electrical Engineering, Columbia University, New York, NY 10027, USA

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Abstract. We use the time-dependent Hartree approximation to obtain the solution to the quantum higher-order non-linear Schrödinger equation. This equation describes femtosecond pulses propagating in non-linear optical fibres and can have soliton solutions. These solitons travel at velocities that differ from the picosecond solitons obtained from the standard quantum non-linear Schrödinger equation. We find that these femtosecond solitons cannot propagate in graded-index fibres; rather, they require quadruple-clad fibres. This is the first investigation of quantum effects in femtosecond solitons to our knowledge.

There is considerable interest in the non-linear Schrödinger equation (NLS) in terms of both classical and quantum phenomena [1–6]. In particular it has been used extensively to model the propagation of pulses in non-linear optical fibres; however, the NLS is generally not valid for pulses with durations shorter than the picosecond time scale. Yet the recent development of optical sources that generate pulses in the femtosecond domain makes possible the exploration of many new phenomena. Therefore the investigation of solitons arising from the higher-order NLS (HNLS), which can be used in the femtosecond time domain, is of interest.

One of the simpletst HNLS is [7]

$$\mathrm{i}\frac{\partial\varphi}{\partial t}+\frac{\partial^2\varphi}{\partial x^2}+2C|\varphi|^2\varphi+\mathrm{i}d\frac{\partial^3\varphi}{\partial x^3}+\mathrm{i}pC|\varphi|^2\frac{\partial\varphi}{\partial x}=0. \tag{1}$$

where C, d and p are constants. We follow the conventional notation in the mathematical literature, which uses t and x to represent normalized space and time, respectively. This equation gives rise to soliton solutions when $p=6d$ [7]. Equation (1) reduces to the NLS for $p=d=0$.

ital (x2)

In certain circumstances the HNLS can be used to describe femtosecond pulses propagating in optical fibers: these are outlined in [8] and described in detail by us in [9]. Using experimental fibre parameters to evaluate the physical parameters in equation (1) we find that the pulse width must be below 200 fs for wavelengths in the 1.48–1.57 μm region in order for d and p to become significant. In addition, the dispersion parameters, β_2 and β_3, given by the second and third derivatives of the propagation constant respectively, evaluated at the carrier frequency ω_0, must be negative. This necessitates a quadruple-clad fibre rather than the typical graded-index fibres used in calculations and experiments to date. This is a significant feature of our results [9]. The soliton self-frequency shift (SSFS) [10, 11] may be an important effect

when considering femtosecond solitons. However, we use the numerical-beam propagation method to show that at distances required for the quantum effect to be observed the effect of the SSFS on the soliton described by equation (1) can be neglected [9].

In the case of optical solitons ϕ represents the normlized envelope of the electromagnetic field. The quantities C and d are given by

$$C \equiv \frac{n_2 \omega_0 \sigma^2 I^2}{c|\beta_2|} \qquad d \equiv \frac{|\beta_3|}{3\sigma|\beta_2|}, \tag{2}$$

and ρ is a parameter involving the frequency-dependent index of refraction and the frequency-dependent radius of the mode of the fibre [8]. n_2 is the non-linear index of refraction, σ is the ~~half~~ width of the pulse ~~duration~~, I is the peak amplitude of the pulse and c is the speed of light.

1/e intensity

solitons

The general solution of equation (1) has the form [7]

$$\phi = \phi_0 \operatorname{sech}[\varepsilon(x - x_0) + \beta t] \exp\{i[\gamma(x - x_0) + \delta t]\}, \tag{3}$$

where ε, β, γ and δ are constants and x_0 is the zero of time. Substituting this in equation (1) yields the following relations

$$|\phi_0|^2 = \varepsilon^2/C \qquad \delta = \varepsilon^2 - \gamma^2 - 3d\gamma\varepsilon^2 + d\gamma^3 \qquad \beta = \varepsilon(2\gamma + d\varepsilon^2 - 3d\gamma^2). \tag{4}$$

We proceed by considering the quantum version of equation (1) from a mathematical point of view. In [9] we examine the physical aspects of this problem in detail and describe the role played by other effects such as the SSFS. The initial portion of our analysis closely follows that of Lai and Haus [5] for the NLS. To obtain the quantum version of equation (1), the quantities $\phi(t,x)$ and $\phi^*(t,x)$ are replaced by the field operators $\hat{\phi}(t,x)$ and $\hat{\phi}^\dagger(t,x)$, which satisfy the bosom commutation relations

$$[\hat{\phi}(t,x'), \hat{\phi}^\dagger(t,x)] = \delta(x - x') \qquad [\hat{\phi}(t,x'), \hat{\phi}(t,x)] = [\hat{\phi}^\dagger(t,x'), \hat{\phi}^\dagger(t,x)] = 0 \tag{5}$$

where $\hat{\phi}(t,x)$ and $\hat{\phi}^\dagger(t,x)$ are the photon annihilation and creation operators, respectively, at t and x.

The quantized equation can be written as

$$i\hbar \frac{\partial}{\partial t} \hat{\phi}(t,x) = [\hat{\phi}(t,x), \hat{H}] \tag{6}$$

with

$$\hat{H} = \hbar\bigg[\int \hat{\phi}_t^\dagger(t,x)\hat{\phi}_t(t,x)\,\mathrm{d}x - C\int \hat{\phi}^\dagger(t,x)\hat{\phi}^\dagger(t,x)\hat{\phi}(t,x)\hat{\phi}(t,x)\,\mathrm{d}x$$
$$+ id\bigg(\int \hat{\phi}_t(t,x)\hat{\phi}_{tt}(t,x)\,\mathrm{d}x - 3C\int \hat{\phi}^\dagger(t,x)\hat{\phi}^\dagger(t,x)\hat{\phi}(t,x)\hat{\phi}_t(t,x)\,\mathrm{d}x\bigg)\bigg] \tag{7}$$

where the subscripts x and xx signify differentiation and double differentiation respectively.

In the Schrödinger picture, the state of the system $|\psi\rangle$ evolves according to

$$\mathrm{i}\hbar \frac{\partial}{\partial t}|\psi\rangle = \hat{H}_s|\psi\rangle \tag{8}$$

where

$$\hat{H}_s = \hbar\Big[\Big[\int \hat{\varphi}_x^{+}(x)\hat{\varphi}_x \,\mathrm{d}x - C\int \hat{\varphi}^{+}(x)\hat{\varphi}^{+}(x)\hat{\varphi}(x)\hat{\varphi}(x)\,\mathrm{d}x$$

(x)

$$+\mathrm{i}d\left(\int \hat{\varphi}_x(x)\hat{\varphi}_{xx}(x)\,\mathrm{d}x - 3C\int \hat{\varphi}^{+}(x)\hat{\varphi}^{+}(x)\hat{\varphi}(x)\hat{\varphi}_x(x)\,\mathrm{d}x\right)\Big]. \tag{9}$$

In general, any state of this system can be expanded in Fock space as

$$|\psi\rangle = \sum_n a_n \int \frac{1}{\sqrt{n!}} f_n(x_1,\ldots,x_n,t)\hat{\varphi}^{+}(x_1)\cdots\hat{\varphi}^{+}(x_n)\,\mathrm{d}x_1\cdots\mathrm{d}x_n|0\rangle. \tag{10}$$

The quantity $|a_n|^2$ is the probability of finding n photons in the field and we require

$$\sum_n |a_n|^2 = 1; \tag{11}$$

f_n obeys the normalization condition

$$\int |f_n(x_1,\ldots,x_n,t)|\,\mathrm{d}x_1\cdots\mathrm{d}x_n = 1. \tag{12}$$

Substituting equations (9) and (10) into (8) we obtain

$$\mathrm{i}\frac{\partial}{\partial t}f_n(x_1,\ldots,x_n,t) = \left(-\sum_{j=1}^{n}\frac{\partial^2}{\partial x_j^2} - 2C\sum_{1\le i<j\le n}\delta(x_j - x_i)\right.$$

$$\left. - \mathrm{i}d\sum_{j=1}^{n}\frac{\partial^3}{\partial x_j^3} - 6\mathrm{i}Cd\sum_{1\le i<j\le n}\delta(x_j - x_i)\frac{\partial}{\partial x_j}\right) f_n(x_1,\ldots,x_n,t). \tag{13}$$

We solve equation (13) using the time-dependent Hartree approximation [12]. We define a Hartree wavefunction

$$f_n^{(H)}(x_1, \ldots, x_n, t) = \prod_{j=1}^{n} \Phi_n(x_j, t). \tag{14}$$

where Φ_n has the normalization

$$\int |\Phi_n(x, t)|^2 \, \mathrm{d}x = 1. \tag{15}$$

The functions Φ_n are determined by minimizing the functional

$$I = \int f_n^{*(H)}(x_1, \ldots, x_n, t) \left[\mathrm{i}\frac{\partial}{\partial t} + \sum_{j=1}^{n} \left(\frac{\partial^2}{\partial x_j^2} + \mathrm{i}d \frac{\partial^3}{\partial x_j^3} \right) \right.$$

$$\left. + \sum_{1 \le i < j \le n} \delta(x_j - x_i) \left(2C + 6\mathrm{i}Cd \frac{\partial}{\partial x_j} \right) \right] f_n^{(H)}(x_1, \ldots, x_n, t) \, \mathrm{d}x_1 \cdots \mathrm{d}x_n. \tag{16}$$

ital

ital

which provides

$$\mathrm{i}\frac{\partial \Phi_n}{\partial t} + \frac{\partial^2 \Phi_n}{\partial x^2} + 2C(n-1)|\Phi_n|^2 \Phi_n + \mathrm{i}d \frac{\partial^3 \Phi_n}{\partial x^3} + 6\mathrm{i}Cd(n-1)|\Phi_n|^2 \frac{\partial \Phi_n}{\partial x} = 0. \tag{17}$$

ital (x2)

This is identical to the classical HNLS given in equation (1), with C replaced by $C(n-1)$. Thus the solution to the quantized femtosecond soliton equation is obtained directly from equations (3) and (4):

$$\Phi_n(x, t) = [C(n-1)]^{-1/2} \varepsilon \, \mathrm{sech}\{\varepsilon[(x - x_0) + (-3d\gamma^2 + d\varepsilon^2 + 2\gamma)t]\}$$

$$\times \exp[-\mathrm{i}(d\gamma^3 + 3d\gamma\varepsilon^2 + \gamma^2 - \varepsilon^2)t + \mathrm{i}\gamma(x - x_0)]. \tag{18}$$

The normalization condition, equation (11), gives

$$\varepsilon = \tfrac{1}{2}(n-1)C. \tag{19}$$

Substituting equation (19) into (18) leads to

$$\Phi_{n\gamma}(x, t) = \tfrac{1}{2}(n-1)^{1/2} C^{1/2} \, \mathrm{sech}\{\tfrac{1}{2}(n-1)C[(x - x_0) + (-3d\gamma^2 + \tfrac{1}{4}d(n-1)^2C^2 + 2\gamma)t]\}$$

$$\times \exp\{[\mathrm{i}d\gamma^3 - \tfrac{3}{4}\mathrm{i}d\gamma(n-1)^2C^2 - \mathrm{i}\gamma^2 + \tfrac{1}{4}\mathrm{i}(n-1)^2C^2]t + \mathrm{i}\gamma(x - x_0)\}. \tag{20}$$

Rom "ez" (x2)

The Hartree product eigenstates are, using equations (10) and (14),

$$|n, \gamma, t\rangle_H = \frac{1}{\sqrt{n!}} \left[\int \Phi_{n\gamma}(x, t) \hat{\phi}^{+}(x) \, \mathrm{d}x \right]^n |0\rangle. \tag{21}$$

A superposition of these states, using a Poissonian distribution of n for a coherent-state pulse, gives

$$|\psi_\gamma\rangle_H = \sum_n \frac{\alpha_0^n}{n!} \mathrm{e}^{-n_0/2} \left(\int \Phi_{n\gamma}(x, t) \hat{\phi}^{+}(x) \, \mathrm{d}x \right)^n |0\rangle. \tag{22}$$

where $|\alpha_0|^2 = n_0$ is the mean photon number.

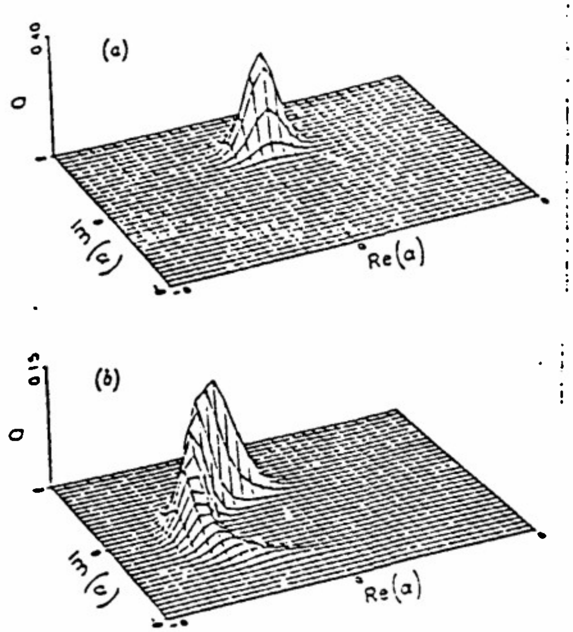

Figure 1. Plots of the quasiprobability density $Q(\alpha, x, t)$ against the real and imaginary parts of α for $\alpha_0 = 4$, $C = 0.25$, $d = 0.25$, $\gamma = 0$, $(x - x_0) = 0$ and (a) $t = 0$ and (b) $t = 0.1$.

The quasiprobability density for the amplitude of the envelope of the field is defined as

$$Q(\alpha, x, t) \equiv |\langle \alpha, x | \psi_t \rangle|^2 \tag{23}$$

where

$$|\alpha, x\rangle \equiv e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} [\hat{\varphi}^{+}(x)]^n |0\rangle \tag{24}$$

is a local coherent state at the time x. Substituting equation (22), with (20) and (24), into (23) gives

superscript

$$\begin{aligned} Q(\alpha, x, t) = {} & e^{-|\alpha|^2-|\alpha_0|^2} \\ & \times \left| \sum_{n=0}^{\infty} \frac{(\alpha^* \alpha_0)^n}{n!} \left(\frac{(n-1)^{1/2}}{2} C^{1/2} \right. \right. \\ & \times \operatorname{sech}\left\{ \frac{n-1}{2} C \left[(x - x_0) + \left(-3d\gamma^2 + d\frac{(n-1)^2}{4} C^2 + 2\gamma \right) t \right] \right\} \Bigg)^n \\ & \times \exp\left[\left(ind\gamma^2 - in(n-1)^2 \tfrac{1}{4} d\gamma C^2 - in\gamma^2 + in\frac{(n-1)^2}{4} C^2 \right) t \right. \\ & \left. \left. + i\gamma n(x - x_0) \right] \right|^2. \end{aligned} \tag{25}$$

In figure 1 we illustrate how this quantity changes as the soliton propagates in space. We have ignored the n dependence of the amplitude and kept it in the phase. We observe phase spreading similar to that in the NLS case [5, 6].

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